Discrete Fourier Transform

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Abstract

A prime factor algorithm is used to compute the Discrete Fourier Transform (DFT) of a complex vector recursively.

1 Introduction

A forward (direct) discrete Fourier transform $y = xW_n^-$ where $W_{n\ jk}^- = e^{-i2\pi jk/n}$ and a reverse (inverse) discrete Fourier transform $y = xW_n^+$ where $W_{n\ jk}^+ = e^{+i2\pi jk/n}$ and $0<n$ is the length of both complex vectors x and y. The computational complexity is of order n^2 if n is a prime number but can be reduced if n can be factored into small prime numbers.

Suppose that $n = pq$ where p is the smallest prime number that can be factored out of n. Then

$$\begin{aligned} y_k &= \sum_{j=0}^{n-1} x_j \cdot e^{\mp i2\pi jk/n} \\ &= \sum_{\ell=0}^{p-1} \sum_{j=0}^{q-1} x_{pj+\ell} \cdot e^{\mp i2\pi(pj+\ell)k/n} \\ &= \sum_{\ell=0}^{p-1} e^{\mp i2\pi\ell k/n} \cdot \sum_{j=0}^{q-1} x_{pj+\ell} \cdot e^{\mp i2\pi jk/q} \qquad (1) \end{aligned}$$

$\forall k \in \{0, 1, \ldots, n-1\}$ and

$$\begin{aligned} y_{qh+k} &= \sum_{\ell=0}^{p-1} e^{\mp i2\pi\ell(qh+k)/n} \cdot \sum_{j=0}^{q-1} x_{pj+\ell} \cdot e^{\mp i2\pi j(qh+k)/q} \\ &= \sum_{\ell=0}^{p-1} e^{\mp i2\pi h\ell/p} \cdot e^{\mp i2\pi\ell k/n} \cdot \sum_{j=0}^{q-1} x_{pj+\ell} \cdot e^{\mp i2\pi jk/q} \qquad (2) \end{aligned}$$

$\forall h \in \{0, 1, \ldots, p-1\}$ and $\forall k \in \{0, 1, \ldots, q-1\}$.

If Y is a $p \times q$ matrix view of vector y where $Y_{hk} = y_{qh+k}$ and X is a $q \times p$ matrix view of vector x where $X_{j\ell} = x_{pj+\ell}$, then the discrete Fourier transform

$$Y = W_p^{\mp} \left(M_n^{\mp} * \left(X^T W_q^{\mp} \right) \right) \tag{3}$$

can be computed recursively by applying the discrete Fourier transform $W_q^{\mp}$ where ${W_q^{\mp}}_{jk} = e^{\mp i2\pi jk/q}$ to each of the p rows of matrix X^T, multiplying matrix $T = X^T W_q^{\mp}$ element by element by the $p \times q$ matrix $M_n^{\mp}$ where ${M_n^{\mp}}_{\ell k} = e^{\mp i2\pi \ell k/n}$ and applying the discrete Fourier transform $W_p^{\mp}$ where ${W_p^{\mp}}_{hl} = e^{\mp i2\pi hl/p}$ to each of the q columns of matrix $M_n^{\mp} * T$.

The transform may be computed in place by transposing matrix X in place recursively before all other processing. If a digit reverse algorithm is used instead of recursive transposition, no element x_j is moved to another location x_k in vector x more than once. Offset

$$j = \sum_{i=0}^{\ell-1} d_{\ell-1-i} \prod_{h=0}^{i-1} r_{\ell-1-h} \tag{4}$$

is computed by first decomposing offset

$$k = \sum_{i=0}^{\ell-1} d_i \prod_{h=0}^{i-1} r_h \tag{5}$$

into digits $0 \le d_i < r_i$ of the mixed radix representation where the ℓ radices are the factors of $n = \prod_{i=0}^{\ell-1} r_i$. In this case, the radices are the prime factors of n in order from least to greatest.

The total number of complex floating-point multiplications C can be calculated from the length

$$n = n_0 = \prod_{j=0}^{\ell-1} p_j \tag{6}$$

of the initial vector and the length

$$n_k = \prod_{j=k}^{\ell-1} p_j \tag{7}$$

of the vectors after k reductions where the p_j are the ℓ prime factors of n.

The total number of complex floating-point multiplications

$$C = C_0 = p_0^2 n_1 + n_0 + p_0 C_1 = n_0 \left(p_0 + 1 + \frac{C_1}{n_1} \right) \tag{8}$$

for the initial function call is calculated by substituting the recursion equation

$$\frac{C_k}{n_k} = p_k + 1 + \frac{C_{k+1}}{n_{k+1}} \tag{9}$$

up through the final equation

$$\frac{C_{\ell-1}}{n_{\ell-1}} = n_{\ell-1} = p_{\ell-1} \tag{10}$$

which yields

$$C = n\left(\sum_{k=0}^{\ell-1}(p_k+1)-1\right). \tag{11}$$

If the prime factors $p_j = p$ are all identical, then

$$C = n\left(\ell\left(p+1\right)-1\right). \tag{12}$$

The computational complexity is $\mathcal{O}\left(n \cdot \log_p(n)\,p\right)$ or $\mathcal{O}\left(n^2\right)$ if $p = n$ or $\mathcal{O}\left(n^{3/2}\right)$ if $p^2 = n$ or $\mathcal{O}\left(n \cdot \lg(n)\right)$ if $p = 2$. The computational complexity of a radix $p > 2$ algorithm is always greater than a radix 2 algorithm by a factor of $p/\lg(p)$ but a radix $p > 2$ implementation may actually be faster than a radix 2 implementation.

2 Real to Complex FFTs

When the source vector x is real, the destination vector y is complex but y_0 is real, $y_{n-k} = y_k^*$ and $y_{n/2}$ is real if n is even. The fast Fourier transform can be computed in-place and returned with the imaginary part of y_0 set to the real part of $y_{n/2}$ if n is even or to the imaginary part of $y_{(n-1)/2}$ if n is odd.

The discrete Fourier transform $W_q^{\mp}$ is applied in-place to each of the p rows of real matrix X^T to form the product $T = X^T W_q^{\mp}$. Because $T_{\ell,0}$ and $T_{\ell,q/2}$ are real, the first $q/2+1$ columns of complex matrix T are packed so that each row of real matrix X^T contains

$$\Re\{T_{\ell,0}\}, \Re\{T_{\ell,q/2}\} \quad T_{\ell,1} \quad T_{\ell,2} \quad \ldots \quad T_{\ell,q/2-1} \tag{13}$$

if q is even but the first $(q+1)/2$ columns of complex matrix T are packed so that each row of real matrix X^T contains

$$\Re\{T_{\ell,0}\}, \Im\{T_{\ell,(q-1)/2}\} \quad T_{\ell,1} \quad T_{\ell,2} \quad \ldots \quad T_{\ell,(q-3)/2} \quad \Re\{T_{\ell,(q-1)/2}\} \tag{14}$$

if q is odd. The transform can be completed for the missing columns of complex matrix T using

$$\begin{aligned}
y_{qh+q-k} &= \sum_{\ell=0}^{p-1} e^{\mp i2\pi\ell h/p} \cdot e^{\mp i2\pi\ell(q-k)/n} \cdot \sum_{j=0}^{q-1} x_{pj+\ell} \cdot e^{\mp i2\pi j(q-k)/q} \\
&= \left(\sum_{\ell=0}^{p-1} e^{\mp i2\pi\ell(p-1-h)/p} \cdot e^{\mp i2\pi\ell k/n} \cdot \sum_{j=0}^{q-1} x_{pj+\ell} \cdot e^{\mp i2\pi jk/q}\right)^* \\
&= \left(\sum_{\ell=0}^{p-1} e^{\mp i2\pi\ell(p-1-h)/p} \cdot M_{n\ \ell k}^{\mp} T_{\ell k}\right)^* = y_{q(p-1-h)+k}^*.
\end{aligned} \tag{15}$$

If both p and q are odd, each of the first $(q+1)/2$ columns of matrix T is multiplied by the corresponding column of matrix $M_n^{\mp}$ element by element then the discrete Fourier transform $W_p^{\mp}$ is applied before the column is stored back into real matrix X^T with row h of complex matrix Y stored in rows $2h$ and $2h+1$ of real matrix X^T

$$\begin{array}{rllll} \Re\{Y_{0,0}\}, \Im\{Y_{0,(q-1)/2}\} & Y_{0,1} & \dots & Y_{0,(q-3)/2} & \Re\{Y_{0,(q-1)/2}\} \\ \Im\{Y_{(p-1)/2,0}\}, \Re\{Y_{0,(q+1)/2}\} & Y_{0,q-1} & \dots & Y_{0,(q+3)/2} & \Im\{Y_{0,(q+1)/2}\} \\ \Re\{Y_{1,0}\}, \Im\{Y_{1,(q-1)/2}\} & Y_{1,1} & \dots & Y_{1,(q-3)/2} & \Re\{Y_{1,(q-1)/2}\} \\ \Im\{Y_{1,0}\}, \Re\{Y_{1,(q+1)/2}\} & Y_{1,q-1} & \dots & Y_{1,(q+3)/2} & \Im\{Y_{1,(q+1)/2}\} \\ & & \vdots & & \\ \Re\{Y_{h,0}\}, \Im\{Y_{h,(q-1)/2}\} & Y_{h,1} & \dots & Y_{h,(q-3)/2} & \Re\{Y_{h,(q-1)/2}\} \\ \Im\{Y_{h,0}\}, \Re\{Y_{h,(q+1)/2}\} & Y_{h,q-1} & \dots & Y_{h,(q+3)/2} & \Im\{Y_{h,(q+1)/2}\} \\ & & \vdots & & \\ \Re\{Y_{(p-3)/2,0}\}, \Im\{Y_{(p-3)/2,(q-1)/2}\} & Y_{(p-3)/2,1} & \dots & Y_{(p-3)/2,(q-3)/2} & \Re\{Y_{(p-3)/2,(q-1)/2}\} \\ \Im\{Y_{(p-3)/2,0}\}, \Re\{Y_{(p-3)/2,(q+1)/2}\} & Y_{(p-3)/2,q-1} & \dots & Y_{(p-3)/2,(q+3)/2} & \Im\{Y_{(p-3)/2,(q+1)/2}\} \\ \Re\{Y_{(p-1)/2,0}\}, \Im\{Y_{(p-1)/2,(q-1)/2}\} & Y_{(p-1)/2,1} & \dots & Y_{(p-1)/2,(q-3)/2} & \Re\{Y_{(p-1)/2,(q-1)/2}\} \end{array} \tag{16}$$

but if p is even and q is odd,

$$\begin{array}{rllll} \Re\{Y_{0,0}\}, \Im\{Y_{0,(q-1)/2}\} & Y_{0,1} & \dots & Y_{0,(q-3)/2} & \Re\{Y_{0,(q-1)/2}\} \\ \Re\{Y_{p/2,0}\}, \Re\{Y_{0,(q+1)/2}\} & Y_{0,q-1} & \dots & Y_{0,(q+3)/2} & \Im\{Y_{0,(q+1)/2}\} \\ \Re\{Y_{1,0}\}, \Im\{Y_{1,(q-1)/2}\} & Y_{1,1} & \dots & Y_{1,(q-3)/2} & \Re\{Y_{1,(q-1)/2}\} \\ \Im\{Y_{1,0}\}, \Re\{Y_{1,(q+1)/2}\} & Y_{1,q-1} & \dots & Y_{1,(q+3)/2} & \Im\{Y_{1,(q+1)/2}\} \\ & & \vdots & & \\ \Re\{Y_{h,0}\}, \Im\{Y_{h,(q-1)/2}\} & Y_{h,1} & \dots & Y_{h,(q-3)/2} & \Re\{Y_{h,(q-1)/2}\} \\ \Im\{Y_{h,0}\}, \Re\{Y_{h,(q+1)/2}\} & Y_{h,q-1} & \dots & Y_{h,(q+3)/2} & \Im\{Y_{h,(q+1)/2}\} \\ & & \vdots & & \\ \Re\{Y_{p/2-1,0}\}, \Im\{Y_{p/2-1,(q-1)/2}\} & Y_{p/2-1,1} & \dots & Y_{p/2-1,(q-3)/2} & \Re\{Y_{p/2-1,(q-1)/2}\} \\ \Im\{Y_{p/2-1,0}\}, \Re\{Y_{p/2-1,(q+1)/2}\} & Y_{p/2-1,q-1} & \dots & Y_{p/2-1,(q+3)/2} & \Im\{Y_{p/2-1,(q+1)/2}\} \end{array} \tag{17}$$

and if $p = 2$ and q is odd,

$$\begin{array}{rllll} \Re\{Y_{0,0}\}, \Im\{Y_{0,(q-1)/2}\} & Y_{0,1} & \dots & Y_{0,(q-3)/2} & \Re\{Y_{0,(q-1)/2}\} \\ \Re\{Y_{1,0}\}, \Re\{Y_{0,(q+1)/2}\} & Y_{0,q-1} & \dots & Y_{0,(q+3)/2} & \Im\{Y_{0,(q+1)/2}\}. \end{array} \tag{18}$$

The real and imaginary parts of complex elements $Y_{h,(q+1)/2}$ through $Y_{h,q-1}$ are stored in reverse order so that the elements in columns 2 through $q-1$ of rows $2h+1$ of real matrix X^T may simply be reversed. The real elements in row $2h$ and column 1 are swapped with the real elements in row $2h+1$ and column 0 of real matrix X^T $\forall h \in \{0, 1, \dots, p/2-1\}$ if p is even or $\forall h \in \{0, 1, \dots, (p-3)/2\}$ if p is odd. Then, if p is odd, the element in row $(p-1)/2$ is swapped with the element in row 0 of column 1 of real matrix X^T.

If both p and q are even, each of the first $q/2+1$ columns of matrix T is multiplied by the corresponding column of matrix $M_n^{\mp}$ element by element then the discrete Fourier transform $W_p^{\mp}$ is applied before row h of complex matrix Y is stored back into rows $2h$ and $2h+1$ of real matrix X^T

$$\begin{array}{rllll}
\Re\{Y_{0,0}\}, \Re\{Y_{0,q/2}\} & Y_{0,1} & Y_{0,2} & \cdots & Y_{0,q/2-1} \\
\Re\{Y_{p/2,0}\}, \Im\{Y_{0,q/2}\} & Y_{0,q-1} & Y_{0,q-2} & \cdots & Y_{0,q/2+1} \\
\Re\{Y_{1,0}\}, \Re\{Y_{1,q/2}\} & Y_{1,1} & Y_{1,2} & \cdots & Y_{1,q/2-1} \\
\Im\{Y_{1,0}\}, \Im\{Y_{1,q/2}\} & Y_{1,q-1} & Y_{1,q-2} & \cdots & Y_{1,q/2+1} \\
 & & & \vdots & \\
\Re\{Y_{h,0}\}, \Re\{Y_{h,q/2}\} & Y_{h,1} & Y_{h,2} & \cdots & Y_{h,q/2-1} \\
\Im\{Y_{h,0}\}, \Im\{Y_{h,q/2}\} & Y_{h,q-1} & Y_{h,q-2} & \cdots & Y_{h,q/2+1} \\
 & & & \vdots & \\
\Re\{Y_{p/2-1,0}\}, \Re\{Y_{p/2-1,q/2}\} & Y_{p/2-1,1} & Y_{p/2-1,2} & \cdots & Y_{p/2-1,q/2-1} \\
\Im\{Y_{p/2-1,0}\}, \Im\{Y_{p/2-1,q/2}\} & Y_{p/2-1,q-1} & Y_{p/2-1,q-2} & \cdots & Y_{p/2-1,q/2+1}
\end{array} \tag{19}$$

and if $p = 2$ and q is even

$$\begin{array}{rllll}
\Re\{Y_{0,0}\}, \Re\{Y_{0,q/2}\} & Y_{0,1} & Y_{0,2} & \cdots & Y_{0,q/2-1} \\
\Re\{Y_{1,0}\}, \Im\{Y_{0,q/2}\} & Y_{0,q-1} & Y_{0,q-2} & \cdots & Y_{0,q/2+1}.
\end{array} \tag{20}$$

Only the first $p/2$ rows of column $q/2$ are stored back into column 1 of real matrix X^T because $Y_{p-1-h,q/2} = Y^*_{h,q/2}$. The real and imaginary parts of complex elements $Y_{h,q/2+1}$ through $Y_{h,q-1}$ are stored in reverse order so that the elements in columns 2 through $q-1$ of rows $2h+1$ of real matrix X^T may simply be reversed. The real elements in row $2h$ and column 1 are swapped with the elements in row $2h+1$ and column 0 of real matrix X^T $\forall h \in \{0, 1, \ldots, p/2-1\}$.

3 Complex to Real FFTs

The complex to real discrete Fourier Transform is computed by reversing the real to complex discrete Fourier Transform.

Equation 1 becomes

$$\begin{aligned}
x_j &= \sum_{k=0}^{n-1} y_k \cdot e^{\mp i2\pi kj/n} \\
&= \sum_{k=0}^{q-1} \sum_{h=0}^{p-1} y_{qh+k} \cdot e^{\mp i2\pi(qh+k)j/n} \\
&= \sum_{k=0}^{q-1} \left(\sum_{h=0}^{p-1} e^{\mp i2\pi jh/p} \cdot y_{qh+k} \right) e^{\mp i2\pi kj/n}
\end{aligned} \tag{21}$$

$\forall j \in \{0, 1, \ldots, n-1\}$.

Equation 2 becomes

$$\begin{aligned} x_{pj+\ell} &= \sum_{k=0}^{q-1} \left(\sum_{h=0}^{p-1} e^{\mp i2\pi(pj+\ell)h/p} \cdot y_{qh+k} \right) e^{\mp i2\pi k(pj+\ell)/n} \\ &= \sum_{k=0}^{q-1} \left(\left(\sum_{h=0}^{p-1} e^{\mp i2\pi \ell h/p} \cdot y_{qh+k} \right) e^{\mp i2\pi \ell k/n} \right) e^{\mp i2\pi kj/q} \end{aligned} \tag{22}$$

$\forall \ell \in \{0, 1, \ldots, p-1\}$ and $\forall j \in \{0, 1, \ldots, q-1\}$.

Equation 3 becomes

$$X = \left(\left(\left(W_p^{\mp} Y \right) * M_n^{\mp} \right) W_q^{\mp} \right)^T . \tag{23}$$

The transform may be computed in place by transposing matrix Y in place recursively after all other processing. If a digit reverse algorithm is used instead of recursive transposition, no element x_k is moved to another location x_j in vector x more than once. Offset

$$k = \sum_{i=0}^{\ell-1} d_i \prod_{h=0}^{i-1} r_h \tag{24}$$

is computed by first decomposing offset

$$j = \sum_{i=0}^{\ell-1} d_{\ell-1-i} \prod_{h=0}^{i-1} r_{\ell-1-h} \tag{25}$$

into digits $0 \le d_i < r_i$ of the mixed radix representation where the ℓ radices are the factors of $n = \prod_{i=0}^{\ell-1} r_i$. In this case, the radices are the prime factors of n in order from least to greatest.